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THE EFFECT OF AN ELECTROSTATIC FIELD ON INVISCID LIQUID FLOW DOWN AN INCLINED PLANE

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Abstract—For a study of nonlinear wave motion under an electrostatic field, the Korteweg-de Vries (KdV) equation for inviscid liquid film flowing under gravity down an inclined plane has been derived, and the effect of the electric field on the stability of a solitary wave as a solution of the KdV equation is examined. Under a constant electrostatic potential the stability of the wave is not affected. However, with a slowly varying potential it becomes unstable.

INTRODUCTION

The study of the dynamics of thin liquid layers is of considerable interest and has application in many engineering processes including film coating, gas absorption, reactors, condensers, evaporators, slot coating, and dip painting. One application of thin liquid films occurs in the design of a space radiator.

Present-day space radiator designs employ armored heat pipes built in by the wick materials such as nickel, copper and titanium, etc. This kind of radiator is then inherently heavy per unit area of radiating surface which can be as much as 20 kg/m^2. Therefore, many advanced radiator concepts have been developed for the design of high performance and lightweight radiators required in space vehicles. For instances, a carbon-carbon heat pipe radiator is proposed by Rovang et al. [1] to make a lightweight radiator. This concept uses 12 independent lithium loops to transfer heat from a thermoelectric generator to heat pipe radiators and a thin metal coating on the inside of the pipe to enhance wettability and prevent material interaction between the base and the heat pipe working fluid. Another type of space radiator was proposed by Al-Baroudi et al. [2]. This is a bubble membrane radiator in which an ellipsoid rotates about the minor axis, vapor is introduced at the center of the ellipsoid and is condensed on the inside surface of the liner. This radiator uses artificial gravity imposed on the working fluid by the centrifugal force to pump the fluid from the radiator. The heat of condensation is conducted through the thin metallic liner which is only required to contain the working fluid. These new types of space radiators will replace the present-day heavy armored radiators. However, the wall of this new radiator has to be made as thin as possible to reduce the radiator weight. Hence, the thinned wall will be vulnerable to punctures by space debris or micrometeorites. These radiators have to be repaired or replaced whenever they are punctured. In this case the maintenance cost will be very high. Therefore, in order to solve this kind of problem an electrostatic liquid film radiator (ELFR) for rejection of heat in space has been already proposed, in which an internal electrostatic field is applied to prevent leakage of the liquid-metal coolant out of any puncture made by the impact of the micrometeorite or the space debris [3-5]. The result is that the ELFR will be allowed to have a thinner armor and have no more frequent repair jobs, hence there will be a considerable weight and maintenance cost savings.

In the previous studies [3-5] the cases of laminar viscous thin-film flows are only considered for the design of ELFR. However, as the fluid velocity becomes large the viscous force of the fluid is dominated by the inertial force and then the fluid can be considered inviscid. For this inviscid film flow with a shallow depth we have examined the nonlinear analysis about a uniform flow and derived a nonlinear evolution equation (KdV equation) with an electrostatic field to investigate a nonlinear wave motion, i.e., to address the question of how the inviscid liquid film and an electrostatic field will interact. Here the pressure dis-

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tribution along the solid wall has not been shown because the same electrostatic strength is used as in the viscous thin-film flow, i.e., where with above 20 kv/cm the pressure at the puncture is sufficiently below the pressure outside the radiator minus the capillary pressure and thus it is always possible to stop a leak of the coolant such as lithium at 700 K through the puncture.

The KdV equation was first derived in 1895 as an asymptotic description of small-but-finite shallow-water waves, where nonlinear and dispersive wave effects interact and dissipative effects can be neglected [6]. The KdV equation arises in all situations which can be approximated by a first order linear hyperbolic system, including a wide variety of physical contexts: magnetohydrodynamics and ion-acoustic plasma [7], etc. The KdV equation is solved exactly for appropriately restricted initial data. There have been several reviews [8, 9] and it is dealt with extensively by Whitham [10] and Newell [11].

Here the KdV equation will be derived with the shallow-water theory for the inviscid film flow down an inclined plane under an electrostatic field. Next the propagation of a solitary wave has been shown numerically with a small amplitude and a long wavelength made by an initial disturbance, in order to examine the effect of an electrostatic field on the film stability.

FORMULATION

Here for the investigation of the effects of the electrostatic field of the inviscid liquid film as it flows down an inclined plane, some comments concerning the static electric field assumption should be made. The electrostatic assumption is valid if $L(\mu_0\varepsilon)^{1/2}/cT \ll 1$, where c is the speed of light, μ_0 is the magnetic permeability, ε is the dielectric constant, L is the characteristic length scale of the disturbance and T is the characteristic unit of time. The value of L may be determined by the geometry, the wavelength of the surface waves on the film interface, or the decay length over which the electric field decays. The value of T is determined by the rate at which the field is turned on or by the frequency of the surface waves. With any of these as the characteristic quantities for an operational ELFR, the electrostatic condition will always be satisfied.

The liquid is considered to be inviscid, incompressible and irrotational. Here the two-dimensional case is only considered. The plane is assumed to make an angle β with the horizontal, and the coordinate system is chosen such that the x-axis is parallel to the plane, while the y-axis is perpendicular to it (see Fig. 1). This implies that the components of the gravitational acceleration in the x and y directions are g sin(β) and $-g$ cos(β), respectively. Above the liquid film there is a vacuum. Within the vacuum region at a distance H from the plane is a charged plate with the same length of the plane wall, which is parallel to the x-axis. To consider this nonlinear wave motion an initial wave disturbance with a small amplitude a and a long wavelength l has been made, and let $x=0$ be at the center of the initial wave (see Fig. 1). Suppose l is the unit of length in the x direction, the inlet film depth h_0 is the unit of length in the y direction and

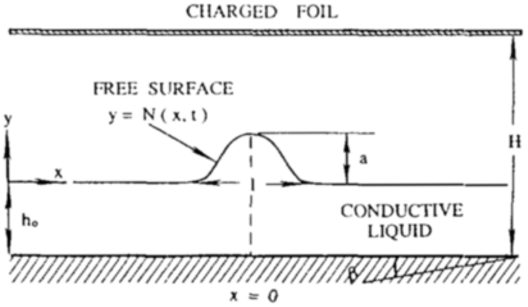

Fig. 1. The coordinate scheme of the solitary wave propagating down an inclined plane with an electrostatic field.

$$\varepsilon=\left(\frac{h_0}{l}\right)^2 \tag{1}$$

and

$$\tilde{\mu}=\left(\frac{a}{h_0}\right), \tag{2}$$

where $\varepsilon \ll 1$, i.e., the horizontal wavelength l is assumed very large when compared to the depth h_0, and $\tilde{\mu} \ll 1$, i.e., the amplitude a is very small compared with the depth h_0. If $h_0/H \ll 1$ then the charged plate is very far from the plane relative to the thickness of the film. Therefore to leading order in the ratio of h_0/H it can be assumed that the charged plate does not see the film and the electrostatic problem for the electric field decouples from the fluid dynamics problem. The ratio $\zeta=H/l$ is assumed to be order one.

The electric field is determined by solving Laplace's equation

$$\nabla^2\phi=0, \tag{3}$$

for the electric potential $\phi(x, y)$ in the fluid, ϕ_v and for the electric potential, ϕ_v, in the vacuum region above the fluid but below the charged plate. For the computational region V, the fluid region, V_f, is defined by $0\le y\le N(x,t)$ and $-L<x<L$, where $y=N(x,t)$ is the height of the film above the inclined plane, and the vacuum region, V_v, is defined by the strip $-L<x<L$ and $N(x,t)\le y\le H-h_0$. We will use the subscript or superscript f for quantities in V_f and the subscript or superscript v for quantities in V_v unless no confusion can occur. The boundary conditions are that

$$\phi(x,y)=FH\Phi(x),\ \text{for}\ y=H-h_0,$$
$$\phi(x,y)=0,\ \text{for}\ y=-h_0. \tag{4}$$

Here F is the characteristic unit of electric field. The function $\Phi(x)$ is a given dimensionless function of x, and the product FH is a constant with the units of electric potential. Along $y=N(x,t)$ the boundary conditions are such that the tangential electric field and the normal displacement field are continuous

$$\phi^f(x,N,t)=\phi^v(x,N,t),\ \varepsilon_f\frac{\partial\phi^f}{\partial n}=\varepsilon_0\frac{\partial\phi^v}{\partial n}. \tag{5}$$

Here ε_f is the dielectric constant of the fluid, ε_0 is the dielectric constant of the vacuum and the partial derivative is in the direction of the outward unit normal, $\mathbf{n}$, to the interface. It should be noted that the interface, $y=N(x,t)$, is unknown, so that the solution of this electrostatic problem is coupled to the dynamics of the film.

The liquid film is governed by the Bernoulli's equation. The velocity potential $\varphi(x,y,t)$ is given by the velocity components $(u,v)=(\partial\varphi/\partial x,\ \partial\varphi/\partial y)$. The independent and dependent variables are scaled as follows:

$$x\to xl,\ y\to h_0\,y,\ H\to h_0\,H,$$
$$N\to a\eta,\ \varphi\to\frac{l}{h_0}a\sqrt{gh_0}\ \ \varphi,\ t\to\frac{l}{\sqrt{g\,h_0}}t,$$
$$\phi\to FH\phi,\ E\to FE. \tag{6}$$

Here g is gravity. With these scales the equations of continuity, the boundary condition on the normal velocity at $y=-h_0$, the continuity of normal stress (pressure) at the free surface, and the kinematic condition are (subscripts denote partial derivatives)

$$\varepsilon\varphi_{xx}+\varphi_{yy}=0, \tag{7}$$

$$\varphi_y=0\ \text{at}\ y=-1, \tag{8}$$

$$\varphi_t+\frac{1}{2}\tilde{\mu}\varphi_x^2+\frac{1}{2}\frac{\tilde{\mu}}{\varepsilon}\varphi_y^2-\frac{x}{\tilde{\mu}\sqrt{\varepsilon}}\sin\beta$$
$$+\eta\cos\beta=K\left[\frac{1}{\varepsilon_f}-1\right][(E_n^v)^2+\varepsilon_f(E_t^v)^2]\ \text{at}\ y=\tilde{\mu}\eta, \tag{9}$$

$$\eta_t+\tilde{\mu}\varphi_x\eta_x=\frac{1}{\varepsilon}\varphi_y\ \text{at}\ y=\tilde{\mu}\eta. \tag{10}$$

Here η is the dimensionless perturbation of $N(x,t)$ from the uniform film height, and the dimensionless constant K is introduced, i.e.,

$$K=\frac{\varepsilon_0F^2}{8\pi ag\rho}, \tag{11}$$

where ρ is the fluid density. The pressure in the vacuum above the liquid film are set to zero. And the scaled Laplace's equation of the electric potential ϕ reduces to

$$\varepsilon\phi_{xx}+\phi_{yy}=0. \tag{12}$$

Also, the dimensionless electric field is defined as

$$\mathbf{E}=\left(\zeta\frac{\partial\phi}{\partial x},\ H\frac{\partial\phi}{\partial y}\right), \tag{13}$$

with the normal component defined as $E_n=\mathbf{E}\cdot\mathbf{n}$ and the tangential component defined as $E_t=\mathbf{E}\cdot\upsilon$ where υ is the unit tangent to the interface. From (9) the charged plate suspended above the liquid film only influences the fluid motion via the inhomogeneous term in the normal stress equation.

The equations are solved by recognizing that (7) admits a power series solution in y and after using (8) it is found,

$$\varphi(x,y,t)=\Gamma(x,t)-\frac{\varepsilon}{2}\Gamma_{xx}(y+1)^2+\frac{\varepsilon^2}{24}\Gamma_{xxxx}(y+1)^4+\cdots, \tag{14}$$

where $\Gamma(x,t)$ is the leading order term of $\varphi(x,y,t)$, i.e., $\varphi_0(x,y,t)$, and the electric potential ϕ can be also expressed as a perturbation expression in ε from (12), i.e.,

$$\phi(x,y,t)=\phi_0(x,y)+\varepsilon\phi_1(x,y,t)+\cdots. \tag{15}$$

Substituting (15) into (12) the ϕ_0 and ϕ_1 in the liquid and gas phases, respectively, are found after using the boundary conditions (4)-(5). The results for the leading order ϕ_0 are

$$\phi_0^f=\Phi(x)\frac{(1+y)}{\varepsilon_f}\left[\frac{1}{\varepsilon_f}+H-1\right]^{-1},\ \text{for}\ -1<y<0,$$
$$\phi_0^v=\Phi(x)\left(y+\frac{1}{\varepsilon_f}\right)\left[\frac{1}{\varepsilon_f}+H-1\right]^{-1},\ \text{for}\ 0<y<H-1, \tag{16}$$

and for ϕ_1

$$\phi_1^f=-\Phi(x)\frac{(1+y)}{\varepsilon_f}\eta\left[\frac{1}{\varepsilon_f}+H-1\right]^{-1},\ \text{for}\ -1<y<0,$$

$$\varphi_1^r = \Phi(x)\left(\frac{y}{(H-1)} - 1\right)\eta\left[\frac{1}{\varepsilon_r} + H - 1\right]^{-1},$$
$$\text{for } 0<y<H-1. \quad (17)$$

Setting $\beta = \varepsilon^{3/2}\beta^*$ where β^* is assumed order one and then substituting (14) into the surface conditions (9)-(10) the following equations are obtained

$$\eta + \Gamma_t + \frac{1}{2}\tilde{\mu}\Gamma_x^2 - \frac{1}{2}(1+\tilde{\mu}\eta)^2\left\{\Gamma_{xxt} + \tilde{\mu}\Gamma_x\Gamma_{xxx} - \tilde{\mu}\Gamma_{xx}^2\right\}\varepsilon$$
$$+ K(E_0^r)^2 + 2\varepsilon K(E_0^r)(E_1^r) + O(\varepsilon^2) = 0 \quad (18)$$

and

$$\eta_t + \{(1+\tilde{\mu}\eta)\Gamma_x\}_x$$
$$-\left\{\frac{1}{6}(1+\tilde{\mu}\eta)^3\Gamma_{xxxx} + \frac{1}{2}\tilde{\mu}(1+\tilde{\mu}\eta)^2\eta_x\Gamma_{xxx}\right\}\varepsilon + O(\varepsilon^2)$$
$$= 0, \quad (19)$$

where the fluid is assumed to have a perfect conductivity ($\varepsilon_r \to \infty$). The leading orders in ε of the derivative of (18) with respect to x and of the Eq. (19) make the following shallow-water equations [10, 11]:

$$\eta_t + \{(1+\tilde{\mu}\eta)u\}_x = 0 \quad (20)$$

and

$$\eta_x + u_t + \tilde{\mu}uu_x = 0, \quad (21)$$

where $u = \Gamma_x$. Finally the normalized form of the KdV equation can be derived from (18) and (19) after using the integration of the $\varphi_x = u - 1/2\varepsilon(y+1)^2u_{xx} + O(\varepsilon^2)$ over the depth from $y = -1$ to $y = \tilde{\mu}\eta$. The normalized form of the KdV equation with the electrostatic field is

$$\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x} + \frac{3}{2}\tilde{\mu}\eta\frac{\partial\eta}{\partial x} + \frac{\partial}{\partial x}\{K(E_0^r)^2\}$$
$$+ \varepsilon\frac{\partial}{\partial x}\left\{\frac{1}{6}\frac{\partial^2\eta}{\partial x^2} + K(E_0^r)(E_1^r)\right\} + O(\tilde{\mu}^2 + \varepsilon^2) = 0. \quad (22)$$

Here the leading-order electric field in ε can be calculated from (16), i.e., $(E_0^r) = \Phi(x)H/(H-1)$ and the first-order electric field $(E_1^r) = \Phi(x)\eta H/(H-1)^2$ from (17).

STABILITY

A linear stability analysis for the longwave approximation in the flow of an incompressible, viscous, thin liquid film down an inclined plane has already been performed [4, 5], obtaining an extension of the classical Yih [12]-Benjamin [13] result for a falling film on an inclined plane wall. The stability is available if

$$Re < \frac{5}{6}\cot(\beta) - \frac{10}{9}K\frac{H^2\left(1-\frac{1}{\varepsilon_r}\right)^2}{\left(\frac{1}{\varepsilon_r} - 1 + H\right)^3}, \quad (23)$$

where Re is the Reynolds number and the second term on the right represents the destabilizing effect of the electrostatic field.

For the classic Korteweg-de Vries equation [6] it is convenient to transform the Eq. (22) at $\Phi = 1$ in a reference frame moving with the basic wave speed $c = 1$ by introducing

$$\xi = x - t, \quad \tau = \frac{1}{6}\varepsilon t. \quad (24)$$

The transformed result is

$$\frac{\partial\eta}{\partial\tau} + 9\frac{\tilde{\mu}}{\varepsilon}\eta\frac{\partial\eta}{\partial\xi} + \frac{\partial^3\eta}{\partial\xi^3} + 6K\frac{\partial}{\partial\xi}$$
$$\left\{\frac{1}{\varepsilon}(E_0^r)^2 + (E_0^r)(E_1^r)\right\} = 0. \quad (25)$$

This equation except for the electrostatic term is Whitham's normal form of the KdV equation [10] and the following solitary wave satisfies (25) at $9\tilde{\mu} = \varepsilon$ without the fourth term:

$$\eta = \mathrm{sech}^2(\xi - 4\tau). \quad (26)$$

The stability of (26) as a solution of (25) without the electric field has been established by Benjamin [14] and Bona [15]. These references deal with one-dimensional perturbations and show that the one-dimensional solitary wave (26) is neutrally stable with respect to small, transverse perturbations.

And considering the linear stability analysis of the Eq. (22) with a time harmonic solution proportional to $exp[i(kx-\omega t)]$, where k is the real wave number and $\omega = \omega_r + i\omega_i$ is the complex frequency, it is found from the Eq. (22) that ω_i is zero, i.e., with this sinusoidal wave disturbance the Eq. (22) is always neutrally stable.

NUMERICAL SOLUTIONS

Here the following KdV equation with an electrostatic field will be solved numerically

$$\frac{\partial\eta}{\partial\tau} + \eta\frac{\partial\eta}{\partial\xi} + \varepsilon\left[\frac{\partial^3\eta}{\partial\xi^3} + 6K\frac{\partial}{\partial\xi}\right.$$
$$\left.\left\{\frac{1}{\varepsilon}(E_0^r)^2 + (E_0^r)(E_1^r)\right\}\right] = 0, \quad (27)$$

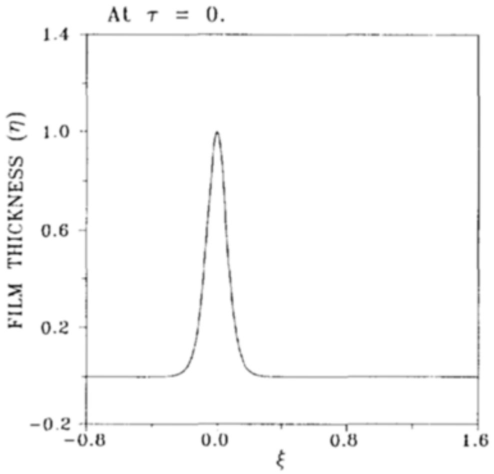

Fig. 2. The initial film thickness of (31).

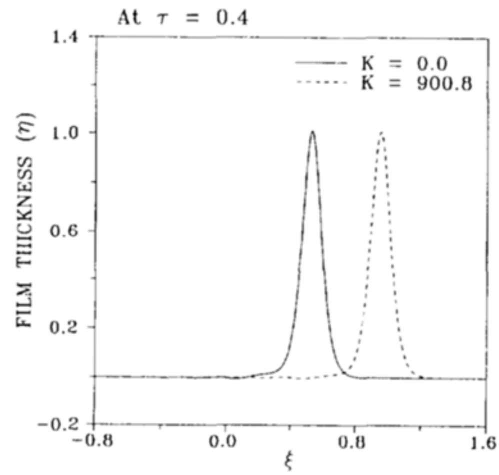

Fig. 3. The film thickness profiles determined by Eq. (33) for $K=0$ (—) and $K=900.8$ (---) at $\tau=0.4$.

where $(E_0{}^r)=\Phi(\xi)H/(H-1)$ and $(E_1{}^r)=\Phi(\xi)\eta H/(H-1)^2$. The above equation without the electric field was formulated by Korteweg-de Vries to explain the solitary waves observed by J. Scott Russell. The Eq. (27) is obtained from the Eq. (22) by setting $\xi=x-t$, $\tau=(1/6)t$ and $\tilde{\mu}=1/9$.

The KdV type Eq. (27) is solved by using the concepts of the usual leap-frog scheme in space [16]:

$$\frac{\partial\eta}{\partial\tau}=\frac{\eta_j^{n+1}-\eta_j^n}{\Delta\tau}, \tag{28}$$

$$\eta\frac{\partial\eta}{\partial\xi}=\frac{1}{3}(\eta_{j+1}^n+\eta_j^n+\eta_{j-1}^n)\frac{\eta_{j+1}^n-\eta_{j-1}^n}{2\Delta\xi}, \tag{29}$$

$$\frac{\partial^3\eta}{\partial\xi^3}=\frac{(\eta_{j+2}^n+\eta_j^n-2\eta_{j+1}^n)-(\eta_j^n+\eta_{j-2}^n-2\eta_{j-1}^n)}{2\Delta\xi^3}, \tag{30}$$

where $\eta_j^n=\eta(j\Delta\xi,\ n\Delta\tau)$.

An initial condition and three boundary conditions are needed to solve (27). The initial condition is

$$\eta(\xi,0)=a\ \mathrm{sech}^2(C_0\xi-C_1) \tag{31}$$

and the boundary conditions $(-L<x<L)$ (Chu et al., 1983) are

$$\eta(-L,\ \tau)=0,\ \eta(L,\ \tau)=\frac{\partial\eta}{\partial\xi}(L,\tau)=0. \tag{32}$$

The initial condition (31) represents a single soliton with amplitude a placed initially at $\xi=(C_1/C_0)$. Here the lithium at 700 K ($\rho=0.493g/cm^3$) is taken as a working fluid.

1. Constant Electrostatic Potential

For a special case we consider $\Phi(\xi)=1$ in the electrostatic potential assuming that the length of the foil is infinite as $L\to\infty$. The Eq. (27) becomes

$$\frac{\partial\eta}{\partial\tau}+\eta\frac{\partial\eta}{\partial\xi}+\varepsilon\left\{\frac{\partial^3\eta}{\partial\xi^3}+6K\frac{\partial\eta}{\partial\xi}\frac{H^2}{(H-1)^3}\right\}=0. \tag{33}$$

The Eq. (33) is solved with the initial condition (31) at $\varepsilon=5.0\times10^{-4}$, $a=1$, $C_0=12.5$ and $C_1=0$ (Fig. 2). At $\tau=0.4$ both results are plotted with an arbitrary electrostatic field ($K=900.8$) and without an electrostatic field ($K=0$) in Fig. 3. The soliton behavior at $K=0$ agrees well with the results obtained by Fornberg and Whitham [17], i.e., there is no dissipation phenomena as the wave moves forward. At $K=900.8$ the amplitude of the wave is unaltered but the wave propagates at a speed depending on the coefficients of $\partial\eta/\partial\xi$, that is, here the solitary wave speeds up by the coefficient $6\varepsilon K[(H^2)/(H-1)^3]$ in (33). The stability of the solitary wave is not affected by the added electrostatic field with $\Phi(\xi)=1$.

2. Slowly varying electrostatic potential in ξ

In order to simulate a slowly varying potential we set $\Phi=0.1\ exp(-100\xi^2)$. With this potential the Eq. (27) reduces to

$$\frac{\partial\eta}{\partial\tau}+\eta\frac{\partial\eta}{\partial\xi}+\varepsilon\frac{\partial^3\eta}{\partial\xi^3}+6K\frac{H^2}{(H-1)^2}\frac{\partial}{\partial\xi}\left[\Phi(\xi)^2\left\{1+\varepsilon\frac{\eta}{H-1}\right\}\right]=0. \tag{34}$$

In Fig. 4 the Eq. (34) is plotted with $K=144.1$ keeping the other parameters as in Fig. 3. The interface is plotted at the times $\tau_n=0.02n$, $n=0,\cdots,15$. The slowly varying potential makes the solitary wave oscillating

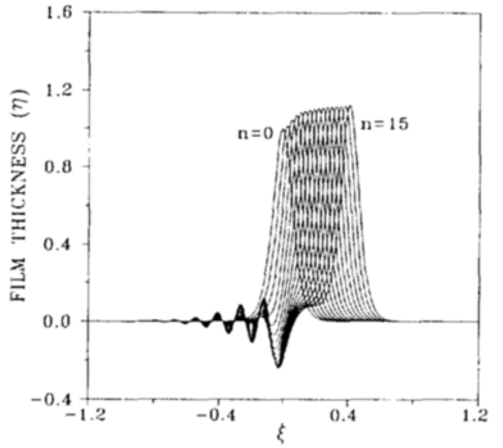

Fig. 4. The film thickness profiles determined by Eq. (34) for $\tau_n = 0.02n$, $n = 0, \cdots, 15$ at $K = 144.1$.

in the upstream as time goes. The solitary wave is unstable.

CONCLUSIONS

The purpose of this investigation is to study the effect of an electrostatic field on an inviscid liquid film flowing down an inclined plane and to develop model equations for this flow system. For a preliminary study the Korteweg-de Vries equation for this system has been derived and the propagation of the solitary waves is examined. The conclusion is that in the considered limit if the electrostatic potential is constant the applied electrostatic field has little influence on the wave stability. This differs from the linear stability analysis in Kim et al. [4, 5], because the plane is nearly horizontal and the effect of the electric field is very small. Also these results will only hold for a finite distance along the plane. If the potential is slowly varying in space, the solitary wave becomes unstable. In this unstable case, a little larger inclination of the angle is needed to increase the film stabilizing effect, i.e., the gravitational component in y-direction. And for the ELFR design, several other problems as in the flow of the viscous liquid film have not been considered here yet. These other questions will be addressed in later works.

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